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SOME NUMERICAL RESULTS ON ASYMPTOTIC FAILURE DISTRIBUTIONS.(U)
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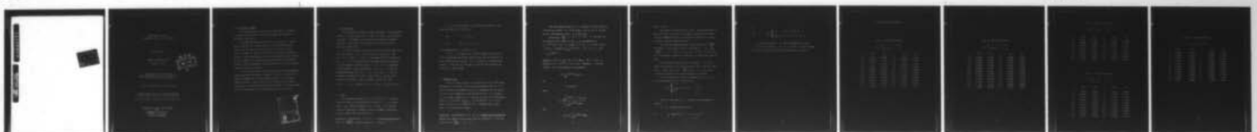

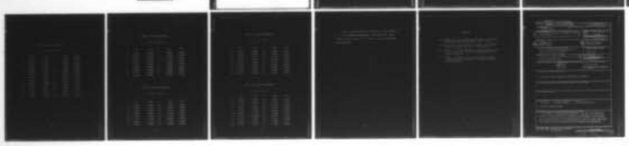

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SOME NUMERICAL RESULTS
ON ASYMPTOTIC FAILURE DISTRIBUTIONS

by

GARY GOTTLIEB

TECHNICAL REPORT NO. 186

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0. Non-Technical Summary

This paper considers a single device shock model. The device is subject to shocks from the outside environment and eventually a shock will cause failure.

This model has been studied in [1], [2], [3] and [4]. The key question asked in all of these papers is of the following nature. Suppose that the ability of the device to survive shocks is such that as the number of shocks that the device has already survived increases, so does the probability that the next shock will cause failure. Then it is asked, if the length of time that the device has survived increases, will the probability that the device fails in the next fixed interval of time also increase.

In [4], this question is considered under much weaker assumptions than in the other three papers. Answers to the type of question cited above are answered only in some approximate sense. In this paper, numerical results are given which help to show how good the approximations are. The key question motivating this paper is how much accuracy is lost if one assumes that the approximations are in fact exact.

1. The Shock Model

In this paper, we consider a simple shock model. The shock model consists of a single device which is subject to shocks from the outside environment. An example is an electrical device which occasionally experiences a large electrical surge due to a malfunction in the electrical supply system.

Each shock may cause the device to fail and eventually some shock will cause failure. Let T be the lifetime of the device. Then $H(t) = P(T \le t)$ $(\bar{H}(t) = 1 - H(t))$ is the time-to-failure distribution. Let $\{Z_t, t \ge 0\}$ be a stochastic process which represents the number of shocks which have arrived by time t. $\{Z_t, t \ge 0\}$ is called the shocking process. We will assume that $\{Z_t, t \ge 0\}$ is an ordinary renewal process and that $P(T > t \mid Z_t) = f(Z_t)$, where f maps the non-negative integers into $[0, 1]$ with $f(0) = 1$ and with $f(n)$ decreasing to zero as n goes to infinity. We let $P_n = f(n)$ $= P(T > t \mid Z_t = n)$. Then $\bar{H}(t) = P(T > t) = EP_{Z(t)} = \sum_{n=0}^{\infty} P(Z_t = n)\, P_n$.

2. Review

This model was studied in [1], [2], [3] and [4]. The authors found conditions on the interarrival distribution F of the renewal process $Z_{(t)}$ and on the sequence $\{P_k\}$ so that H, the time-to-failure distribution, is related to one of the common distributional notions of aging (IFR, NBU, IFRA, etc.).

Definition: A distribution H on $[0, \infty)$ is Increasing Failure Rate (IFR) if $\frac{1-H(t+x)}{1-H(t)}$ is non-increasing in t for any $x > 0$.

If T has a distribution H, an equivalent statement of the above definition is: H is IFR if

(1) $$T \geq 0 \text{ a.s.}$$

(2) $$P(T > t + x \mid T > t)$$

is non-increasing in t for any $x > 0$.

In [1], Z_t was taken to be a Poisson renewal process, in [2] to be a non-homogeneous Poisson renewal process and in [3] and [4] to be an ordinary renewal process. In [1], [2] and [3], conditions were found on F and the $\{P_k\}$ sequence so that the time-to-failure distribution H would be IFR.

3. Asymptotic Case

In [1], [2] and [3], when H was shown to be IFR, it was always assumed that F was IFR. In [4], the only assumption made on F is that its Laplace transformation exists in a neighborhood of the origin, the size of the neighborhood being determined by the $\{P_k\}$ sequence. With this weaker assumption on F, and with the same sort of assumptions made on P_k as before (P_{k+1}/P_k non-increasing in k), it is no longer true that H is necessarily IFR. A somewhat weaker result about H will often hold, though.

Definition: A distribution H on $[0, \infty)$ is Asymptotically Increasing Failure Rate (AIFR) if there exists some IFR distribution G with the property that $\frac{1-H(u)}{1-G(u)} \to 1$ as $u \to \infty$.

Under this weaker assumption on F (Laplace transform existence), conditions are found in [4] on the sequence $\{P_k\}$ so that H is AIFR. Further, the asymptotic form of H is specified.

Suppose that $\frac{P_{k+1}}{P_k} \downarrow e^{-\gamma}$ as $k \to \infty$ with $\gamma > 0$ and that the convergence is quick enough so that P_k can be expressed as $P_k = e^{-\gamma k} L(k) k^{\rho}$ for $k \geq 0$, where $\rho \geq 0$ and L is slowly varying at infinity. Suppose that $\int_{x=0}^{\infty} e^{sx} F(dx) < \infty$ for $(x) \leq \gamma/\mu$ where $\mu = \int_{x=0}^{\infty} xF(dx) < \infty$.

<u>Theorem 1</u>. Let H, F and $\{P_k\}$ be as above. Then $1 - H(t)$ is asymptotic to $K \cdot e^{-\phi(\gamma)t} t^{\rho} L(t) = \psi(t)$. Therefore, H is AIFR. Further, $\phi(\gamma)$ is the solution to the equation

$$e^{-\gamma} \int_{s=0}^{\infty} e^{\phi(\gamma)s} F(ds) = 1,$$

and

$$K = c_{\gamma}(1/\mu_G)^{\rho}$$

where

$$c_{\gamma} = \frac{\int_{s=0}^{\infty} e^{\phi(\gamma)s} (1 - F(s))\, ds}{e^{-\gamma} \int_{s=0}^{\infty} se^{\phi(\gamma)s} F(ds)}$$

and

$$\mu_G = e^{-\gamma} \int_{s=0}^{\infty} se^{\phi(\gamma)s} F(ds) .$$

Proof. See [4].

This result is potentially very useful. We may approximate H, which is usually very difficult to compute, by a distribution which is easily computable and has a form which will often make it easy to determine an optimal replacement policy.

The important remaining question is how quickly does $\frac{1-H(t)}{\psi(t)}$ approximate 1 and crucially, what is the value of $1 - H(t)$ when $\left| \frac{1-H(t)}{\psi(t)} - 1 \right| < \epsilon$, where ϵ is say .05. If $H(t)$ is still near 0, the approximation suggested by Theorem 1 is of significant practical value.

We will present numerical results in Tables 1 - 10 for two simple distributions F. The tables will list $\bar{H}(t)$ and $R(t)$, where $\bar{H}(t) = 1 - H(t)$ and $R(t) = c \cdot \frac{\bar{H}(t)}{\psi(t)}$ *and* $c > 0$. *In each case*, according to Theorem 1, $R(t)$ should converge to a constant.

F will either be the uniform distribution on $[0, 1]$ or a truncated exponential of the following form:

$$F(t) = \begin{cases} 0 & t \leq 0 \\ \frac{1}{1-e^{-1.5}} (1 - e^{-1.5t}) & t \in (0, 1] \\ 1 & t > 1 \end{cases} .$$

Clearly, either choice of F satisfies the requirements of Theorem 1.

The $\{P_k\}$ sequence will be either

$$(1) \qquad P_k = \min(e^{-\gamma k} k^{\rho}, 1) \qquad e^{-\gamma} = .7, .8, .9$$
$$\rho = 1, 2$$

or

$$(2) \qquad P_k = \begin{cases} 1 & k \le n \quad n = 10,\ 20 \\ e^{-\gamma(k-n)} & k > n \quad e^{-\gamma} = .7,\ .8,\ .9\ . \end{cases}$$

For both (1) and (2), P_k can be expressed in the form $P_k = e^{-\gamma k} k^{\rho} L(k)$, where $\gamma > 0$, $\rho > 0$ and $L(k)$ is slowly varying at infinity, therefore satisfying the conditions of Theorem 1.

Table 1 - Uniform Distribution

$$P_k = \begin{cases} 1 & k \leq 10 \\ (.9)^{k-10} & k > 10 \end{cases}$$

T	$\bar{H}(T)$	R(T)	T	$\bar{H}(T)$	R(T)
0.0	1.00000	0.10000	1.0	0.99838	0.12282
2.0	0.99834	0.15108	3.0	0.99648	0.18551
4.0	0.97616	0.22355	5.0	0.89891	0.25324
6.0	0.76516	0.26518	7.0	0.62612	0.26693
8.0	0.50884	0.26687	9.0	0.41337	0.26669
10.0	0.33575	0.26647	11.0	0.27264	0.26620
12.0	0.22135	0.26585	13.0	0.17964	0.26542
14.0	0.14574	0.26489	15.0	0.11818	0.26423
16.0	0.09577	0.26342	17.0	0.07755	0.26242
18.0	0.06275	0.26118	19.0	0.05071	0.25965
20.0	0.04092	0.25776	21.0	0.03296	0.25544

Table 2 - Uniform Distribution

$$P_k = \begin{cases} 1 & k \leq 20 \\ (.9)^{k-20} & k > 20 \end{cases}$$

T	$\bar{H}(T)$	R(T)	T	$\bar{H}(T)$	R(T)
0.0	1.00000	1.00000	1.0	0.99980	1.22992
2.0	0.99980	1.51301	3.0	0.99980	1.86125
4.0	0.99980	2.28965	5.0	0.99980	2.81664
6.0	0.99968	3.46452	7.0	0.99823	4.25579
8.0	0.98904	5.18712	9.0	0.95531	6.16338
10.0	0.87879	6.97467	11.0	0.76320	7.45152
12.0	0.63544	7.63208	13.0	0.51929	7.67266
14.0	0.42240	7.67750	15.0	0.34337	7.67752
16.0	0.27911	7.67706	17.0	0.22687	7.67637
18.0	0.18439	7.67538	19.0	0.14987	7.67403
20.0	0.12180	7.67226	21.0	0.09898	7.66994
22.0	0.08043	7.66898	23.0	0.06535	7.66327

Table 3 - Uniform Distribution

$$P_k = \begin{cases} 1 & k \leq 10 \\ (.8)^{k-10} & k > 10 \end{cases}$$

T	$\bar{H}(T)$	R(T)	T	$\bar{H}(T)$	R(T)
0.0	1.00000	0.10000	1.0	0.99989	0.15384
2.0	0.99982	0.23667	3.0	0.99622	0.36282
4.0	0.95798	0.53679	5.0	0.81858	0.70571
6.0	0.59585	0.79034	7.0	0.39532	0.80674
8.0	0.25723	0.80766	9.0	0.16720	0.80771
10.0	0.10867	0.80767	11.0	0.07061	0.80747
12.0	0.04587	0.80705	13.0	0.02979	0.80628
14.0	0.01933	0.80496	15.0	0.01253	0.80281

Table 4 - Uniform Distribution

$$P_k = \begin{cases} 1 & k \leq 10 \\ (.7)^{k-10} & k > 10 \end{cases}$$

T	$\bar{H}(T)$	R(T)	T	$\bar{H}(T)$	R(T)
0.0	1.00000	0.01000	1.0	1.00000	0.01965
2.0	0.99989	0.03861	3.0	0.99468	0.07547
4.0	0.94058	0.14022	5.0	0.75097	0.21999
6.0	0.47001	0.27055	7.0	0.25019	0.28298
8.0	0.12775	0.28393	9.0	0.06506	0.28413
10.0	0.03313	0.28433	11.0	0.01687	0.28452
12.0	0.00859	0.28471	13.0	0.00438	0.28489
14.0	0.00223	0.28506	15.0	0.00113	0.28520
16.0	0.00058	0.28529	17.0	0.00029	0.28527
18.0	0.00015	0.28505	19.0	0.00008	0.28443

Table 5 - Uniform Distribution

$$P_k = \min(1, (.8)^k \cdot k)$$

T	$\bar{H}(T)$	R(T)	T	$\bar{H}(T)$	R(T)
0	1.00000	0.99455	1	1.00000	0.76508
2	1.00000	0.78904	3	0.99984	0.91034
4	0.99616	1.11637	5	0.96769	1.39041
6	0.87275	1.85374	7	0.70970	1.81039
8	0.53523	1.86721	9	0.39070	1.88734
10	0.28125	1.90031	11	0.20055	1.91105
12	0.14189	1.92021	13	0.09973	1.92815
14	0.06970	1.93610	15	0.04848	1.94131
16	0.03367	1.94682	17	0.02316	1.95178
18	0.01593	1.95628	19	0.01092	1.96039
20	0.00747	1.96415	21	0.00509	1.96763

Table 6 - Uniform Distribution

$$P_k = \min(1, (.8)^k \cdot k^2)$$

T	$\bar{H}(T)$	R(T)	T	$\tilde{H}(T)$	R(T)
0	1.00000	0.77656	1	1.00000	0.29869
2	1.00000	0.26301	3	1.00000	0.22762
4	1.00000	0.22413	5	1.00000	0.23947
6	1.00000	0.27069	7	1.00000	0.31886
8	1.00000	0.38762	9	1.00000	0.48307
10	0.99997	0.61422	11	0.99976	0.79391
12	0.99828	1.03923	13	0.99126	1.36896
14	0.96784	1.79140	15	0.91147	2.28131
16	0.81094	2.76622	17	0.67400	3.15517
18	0.52580	3.39889	19	0.39180	3.51675
20	0.28445	3.56304	21	0.20397	3.58156
22	0.14534	3.59242	23	0.10311	3.60140
24	0.07289	3.60962	25	0.05135	3.61736
26	0.03606	3.62453	27	0.02525	3.63123
28	0.01764	3.63751	29	0.01229	3.64338

Table 7 - Uniform Distribution

$$P_k = \min(1, (.7)^k \cdot k)$$

T	$\bar{H}(T)$	R(T)	T	$\bar{H}(T)$	R(T)
0	1.00000	0.99997	1	0.98917	0.98215
2	0.98031	1.26170	3	0.84602	1.60468
4	0.59408	1.77134	5	0.37382	1.82512
6	0.22607	1.85901	7	0.13331	1.88473
8	0.07714	1.90504	9	0.04400	1.92162
10	0.02481	1.93541	11	0.01386	1.94714
12	0.00768	1.95728	13	0.00423	1.96617
14	0.00231	1.97410	15	0.00126	1.98118
16	0.00068	1.98761	17	0.00037	1.99347

Table 8 - Truncated Exponential

$$P_k = \min(1, (.7)^k \cdot k)$$

T	$\bar{H}(T)$	R(T)	T	$\bar{H}(T)$	R(T)
0.0	1.00000	1.00000	0.5	0.99917	1.08820
1.0	0.99445	1.17957	1.5	0.96562	1.44042
2.0	0.88636	1.66277	2.5	0.75445	1.88786
3.0	0.59885	1.99881	3.5	0,45269	2.08157
4.0	0.33253	2.10643	4.5	0.24007	2.13822
5.0	0.17112	2.14298	5.5	0.12069	2.15523
6.0	0.08425	2.14529	6.5	0.05832	2.13976
7.0	0.03990	2.10928	7.5	0.02707	2.07809

Table 9 - Truncated Exponential

$$P_k = \min(1,\ (.8)^k \cdot k)$$

T	$\bar{H}(T)$	R(T)	T	$\bar{H}(T)$	R(T)
0.0	1.00000	1.00000	0.5	0.96162	0.89831
1.0	0.96162	0.83916	1.5	0.96161	0.90516
2.0	0.96138	0.97613	2.5	0.95970	1.11485
3.0	0.95256	1.26600	3.5	0.93201	1.46367
4.0	0.88875	1.64923	4.5	0.81796	1.83054
5.0	0.72365	1.95308	5.5	0.61697	2.03663
6.0	0.51038	2.06067	6.5	0.41280	2.05962
7.0	0.32825	2.02395	7.5	0.25729	1.97592
8.0	0.19875	1.90109	8.5	0.15097	1.80986

Table 10 - Truncated Exponential

$$P_k = \min(1,\ (.7)^k \cdot k^2)$$

T	$\bar{H}(T)$	R(T)	T	$\bar{H}(T)$	R(T)
0.0	1.00000	1.00000	0.5	0.97834	0.75343
1.0	0.97834	0.58023	1.5	0.97834	0.59579
2.0	0.97834	0.61177	2.5	0.97832	0.70669
3.0	0.97818	0.81623	3.5	0.97728	1.00483
4.0	0.97358	1.23344	4.5	0.96226	1.56475
5.0	0.93545	1.95243	5.5	0.88448	2.43716
6.0	0.80453	2.92668	6.5	0.69851	3.42453
7.0	0.57707	3.81287	7.5	0.45435	4.11004
8.0	0.34258	4.24273	8.5	0.24882	4.27171
9.0	0.17488	4.16178	9.5	0.11907	3.96760

Clearly, the best results are in Tables 4 - 6. The convergence of H to its asymptotic approximation is relatively quick. Additionally, the closer that $e^{-\gamma}$ is to one or ρ to zero, the better the approximation.

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We consider a single device shock model. The device is subject to shocks from the environment and eventually a shock will cause the device to fail. In an earlier paper, an asymptotic approximation for the time-to-failure distribution of the device was given. ~~In~~ this paper, ~~we~~ computes numerically the actual time-to-failure distribution for various cases and then compare these distributions with the asymptotic approximations.

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